flow region exists in real rocket motors will depend on factors such as initial conditions, grain geometry, and even igniter gas flow characteristics. However, in the case of high performance rocket motors, we believe that strong convective forces inside a rocket motor establish a boundary-layer flow over the propellant surface for which both developing boundary-layer and the potential-core regions exist. As far as the experimental evidence on the nature of the flowfield cited by King, it should be noted that these experiments were conducted under nonreactive "simulative" flow conditions. But the boundary layer in an actual grain port may be different due to differences in temperature, pressure, chemical reactions, etc.

Furthermore, we have reviewed the results of the simulative study by Yamada et al.[4] and find no evidence or conclusion in regard to the nonexistence of the potential core. In boundary-layer flows, it is well known that turbulence is produced mainly by mean shear in the near-wall region. This was also observed in the experiments of Ref. 4. Yamada et al. further point out that the role of turbulence adjacent to a propellant surface is to enhance mixing rate of decomposed gases and increase the heat transfer rate. This is precisely what our model predicts and is also part of the erosive burning mechanism suggested by our study. It may be noted that our model can take into account a nonzero freestream turbulence level (which may be present as a result of initial conditions such as igniter gas flow characteristics) and its spread within the boundary layer through the boundary conditions, Eqs. (27) in Ref. 1.

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Comment on "Finite Elements for Initial Value Problems in Dynamics"

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DR. Simkin's paper[1] is quite significant. In my opinion, it indicates that he has given additional thought to the problem of direct solutions to time dependent systems.[2,3] Others,[4-6] through application of Hamilton's Law of Varying Action (HLVA), have obtained gratifying and significant results. Dr. Simkins references certain of my papers and then invokes d'Alembert's Principle as his starting point.[1] But my starting point was never d'Alembert's Principle nor any other force or energy balance equation.

D'Alembert's Principle, as well as his ideas about virtual work, were presented about 240 years ago. At about the same time, however, other ideas and concepts which have influenced the development and application of the truths of mathematics to mechanics and physics were set forth. As implied in my Comment[3] on Dr. Simkins' previous paper,[2] and as obviously now recognized by Dr. Simkins[1] and others,[7] the problem lies in what we have all been taught about Hamilton's Principle. The confusion[8] which has been apparent for years and which has now been brought into the open literature,[9-11] can be traced directly to the foundational concepts of the Calculus of Variations.[12]

Examine the Fundamental Lemma of the Calculus of Variations:

"If

$$\int_{x_1}^{x_2} M\eta \mathrm{d}x = 0$$

for all functions η which vanish at x_1 and x_2 and poses a continous derivative on (x_1, x_2) then $M=0$ on (x_1, x_2)."

This Fundamental Lemma constitutes an unquestionable mathematical truth. It transcends even that branch of mathematics called the Calculus of Variations. Note that the function $M=0$ is both necessary and sufficient for the definite integral to vanish when the function η satisfies the prescribed conditions. Further note that the function M must be some sort of "balance" or "conservation" equation which must be known to vanish, a priori, because it cannot itself be determined from this fundamental mathematical truth. D'Alembert's Principle, $F-\dot{P}=0$, is a force balance equation, directly from Newton's Law, which qualifies as a function M. Also note that the Fundamental Lemma is exactly of the form obtained by application of the Galerkin method to any differential balance or conservation type of equation which is known, a priori, to vanish.

Dr. Simkins integrates d'Alembert's Generalized Principle of Virtual Work over an interval of time, $t_1 > t_2$,

$$\sum_{i=1}^{N} \int_{t_1}^{t_2} (F_i - \dot{P}_i)\delta r_i \mathrm{d}t = 0 \tag{1}$$

This equation is seen to be exactly of the form required by the Fundamental Lemma of the Calculus of Variations. Integrate by parts and one does, indeed, obtain a form of Hamilton's Law of Varying Action, which (except for one term) was presented by Hamilton 147 years ago,

$$\sum_{i=1}^{N} \left[\int_{t_1}^{t_2} (F_i\delta r_i + P_i\delta \dot{r}_i)\mathrm{d}t - P_i\delta r_i \Big|_{t_1}^{t_2} \right] = 0 \tag{2}$$

There should be no argument with what Dr. Simkins has done *except* for the foundational concepts associated with the Calculus of Variations. *If* the Fundamental Lemma requires that the function, η, vanish at x_1 and x_2 *before* integration by parts, logically the functions δr_i must vanish at t_1 and t_2 *after* integration by parts. Thus, one obtains Hamilton's *Principle*

$$\sum_{i=1}^{N} \int_{t_1}^{t_2} (F_i\delta r_i + P_i\delta \dot{r}_i)\mathrm{d}t = 0 \tag{3}$$

It is now clear that this equation and the concepts associated therewith have caused massive confusion with respect to direct solutions to the problems of mechanics and physics[7-15] in both the time and space domains.

If one wishes to associate "potential" functions with a physical system, d'Alembert's Generalized Principle of Virtual Work, integrated over time, again yields exactly HLVA,

$$\delta \int_{t_1}^{t_2} (T-V)\mathrm{d}t + \int_{t_1}^{t_2} \sum_{i=1}^{N} F_i\delta r_i \mathrm{d}t - \sum_{i=1}^{N} P_i\delta r_i \Big|_{t_1}^{t_2} = 0 \tag{4}$$

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But again, if one is to adhere to the Fundamental Lemma of the Calculus of Variations, the functions δr_i must vanish at t_1 and t_2. Thus, one arrives at the extended form of Hamilton's *Principle* for a nonconservative system,

$$\delta \int_{t_1}^{t_2} (T-V)\,\mathrm{d}t + \int_{t_1}^{t_2} \sum_{i=1}^{N} F_i \delta r_i \mathrm{d}t = 0 \qquad (5)$$

One reviewer of Ref. 13 pointed out that Eq. (5) is "... a well known equation in any decent textbook on nonconservative problems." Unfortunately, this reviewer is totally correct. After some further discussion, this same reviewer suggested the first sentence of the conclusion to Ref. 13: "The vibration and stability of follower force systems have been demonstrated through a theory based on Hamilton's Law of Varying Action *hitherto not found in the literature*" (emphasis mine).

Still another reviewer[14] has written that "... Dr. Bailey has the merit to have pointed out that what he calls Hamilton's law has been wrongly mutilated by ignorants" (These are the reviewer's words. They are not mine). But he is partially correct. I believe the important question is, "*Why* was Hamilton's law 'mutilated'?" It was not done by "ignorants."

Dr. Simkins correctly writes about Hamilton's Principle, Eqs. (3) or (5), "Thus Eq. (3) cannot, with complete logic, be used to formulate any system of initial value problems of dynamics." Dr. Simkins has perceived a truth which, except for the foundational concepts of the Calculus of Variations, should have and, I believe, would have been perceived and become common knowledge 200 years ago.[15]

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